

Lie Rinehart Bialgebras for Crossed Products

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Abstract

In this paper, we study Lie Rinehart bialgebras, the algebraic generalization of Lie bialgebroids. More precisely, we analyze the structure of Lie Rinehart bialgebras for crossed products induced by actions of Lie algebras on $K[t]$.

Key words Lie Rinehart algebras, Lie Rinehart bialgebras, Crossed products, actions of Lie algebras.

MSC Primary 17B62, Secondary 17B70.

Introduction

The concept of Lie Rinehart algebras was introduced in [25] as an abstract algebraic treatment of the category of Lie algebroids [8, 20] and were investigated further in many texts [1, 9–11]. For the details and history of the notion of Lie Rinehart algebras, one may refer to an expository paper of Huebschmann [12]. Lie Rinehart algebras may be seen as an algebraic generalization of the notion of Lie algebroid in which the space of sections of the vector bundle is replaced by a module over a ring, vector fields by derivations of the ring and so on. Any attempt to extend Lie algebroid theory to singular spaces leads to Lie Rinehart algebras. The reader interested in Lie algebroids and groupoids is referred to Mackenzie's new book [21] (see also [2] [Chapters 8 and 12] [19] [I and III]) for background information.

In this paper, we always assume that \mathbf{A} is a commutative, associative algebra over K (\mathbb{R} or \mathbb{C}) with a unit. A Lie Rinehart algebra is an \mathbf{A} -module that admits a K -Lie bracket and an action on \mathbf{A} (called anchor), which are compatible in a certain sense (see Definition 1.1).

The notion of Lie Rinehart bialgebras is first introduced by Huebschmann [13]. It is derived from the notion of Lie bialgebras, introduced by Drinfel'd in [5], and the notion of Lie bialgebroids introduced by Mackenzie and Xu in [22] as the infinitesimal objects associated to a Poisson groupoid. Lie algebroids are generalized tangent bundles, while Lie bialgebroids can be considered as generalizations of both Poisson structures and Lie bialgebras. Roughly speaking, a Lie bialgebroid is a Lie algebroid A whose dual A^* is also equipped with a Lie algebroid structure, which is compatible in a certain sense with that of A . This compatibility condition can be expressed equivalently in terms of the pair (A, d_*) , where $d_* : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+1} A)$ is the differential operator inducing the Lie algebroid structure on A^* . Analogously, a Lie Rinehart bialgebra (E, σ) is a Lie Rinehart algebra E endowed with a graded operator $\sigma : \wedge_{\mathbf{A}}^k E \rightarrow \wedge_{\mathbf{A}}^{k+1} E$ satisfying $\sigma^2 = 0$ and a condition similar to Drinfel'd's cocycle condition (see Definition 1.8). A similar concept, namely generalized Lie bialgebras, was defined in [26], where the two Lie algebras are not dual in the usual sense.

In [5], Drinfel'd classified Lie bialgebras successfully. The classification of Lie Rinehart bialgebras is, in our humble opinion, more challenging. However, if we restrict our attention to the special class of crossed products (first introduced by Malliavin in [23]), we can get lots of information about their algebraic structure and Lie Rinehart bialgebra structures over them. The correspondence of crossed products of Lie algebroids are known as action Lie algebroids or transformation Lie algebroids.

The purpose of this paper is twofold. First, we investigate the structure of actions of a Lie algebra on the ring of polynomials $K[t]$ and we describe the special features of a $K[t]$ -crossed product¹. Second, we classify all possible Lie Rinehart bialgebras $(K(t) \otimes \mathfrak{g}, d_*)$ ($K(t)$ denoting the fraction field of $K[t]$) in which $K(t) \otimes \mathfrak{g}$ is a crossed product coming from an action of \mathfrak{g} on $K[t]$. It turns out that our classification is very similar to the results of [18] and [3], i.e. the operator d_* (which determines the dual Lie Rinehart algebra structure) is the sum $[\Lambda, \cdot] + \Omega$ of a bivector Λ and some cocycle Ω . As in [3], we call the data (Λ, Ω) a compatible pair. In the particular case that \mathfrak{g} is semisimple, the Lie Rinehart bialgebra structure is related to the so-called ε -dynamical r -matrices. In fact, it is a special case of the dynamical r -matrices coupled with Poisson manifolds introduced in [18].

The paper is organized as follows.

Section 1 reviews Lie Rinehart algebras, Lie Rinehart bialgebras and crossed products. Most importantly, we recall some fundamental properties of the

¹In this paper we only deal with actions of Lie algebras on $K[t]$. But in many situations we have to consider the fraction field $K(t)$. We expect to get a classification of Lie algebra actions on $K(t)$ and further results concerning Lie Rinehart bialgebras.

Schouten bracket and Gerstenhaber algebras.

Section 2 builds on the foundations laid forth in [4], namely the classification of actions of a finite dimensional Lie algebra on $K[t]$. Its main result is Theorem 2.4, which asserts that any $K[t]$ -crossed product is an extended one.

Section 3 is devoted to the classification of Lie Rinehart coalgebras for $K[t]$ -crossed products. The main results are Theorems 3.3 and 3.5, which can be summarized as follows. If $K[t] \otimes \mathfrak{g}$ is nontrivial, then the differential operator of any bialgebra $(K(t) \otimes \mathfrak{g}, d_*)$ decomposes as $d_* = [\Lambda, \cdot] + \Omega$, where Λ is a bivector of $K(t) \otimes \mathfrak{g}$ and Ω is a map from \mathfrak{g} to $L^2 = L \wedge_{K(t)} L$ with L denoting the kernel of $\theta : K(t) \otimes \mathfrak{g} \rightarrow K(t)$. The data (Λ, Ω) is called a compatible pair of $K(t) \otimes \mathfrak{g}$.

Section 4 details the special properties enjoyed by the data (Λ, Ω) in the particular case that \mathfrak{g} is a semi-simple Lie algebra. Notice that, in this case, any nontrivial action of \mathfrak{g} merely comes from $\mathfrak{sl}(2, K)$, which must be an ideal of \mathfrak{g} . The conclusion is that the corresponding Lie Rinehart bialgebras $(K[t] \otimes \mathfrak{g}, d_*)$ can be characterized by an ε -dynamical r -matrix Λ such that $d_* = [\Lambda, \cdot] + \varepsilon \mathcal{D}$. Here $\mathcal{D} : K[t] \otimes \wedge^k \mathfrak{g} \rightarrow K[t] \otimes \wedge^{k+1} \mathfrak{g}$ is a fixed operator and ε is a constant number in K (see Definition 4.3 and Theorem 4.4). In this case, the data $(\Lambda + \varepsilon \tau, \varepsilon(\mathcal{D} - [\tau, \cdot]))$ is a compatible pair, where $\tau \in K[t] \otimes \wedge^2 \mathfrak{sl}(2, K)$ is an ε -dynamical r -matrix with $\varepsilon = -1$ (see Proposition 4.10).

Acknowledgments Besides CPSF(20060400017), the present work was completed while Zhuo Chen was visiting the mathematics department at Penn State with support from its Shapiro fund. D.-S. Zhong is grateful to Hongkong University for its hospitality. The authors thank Johannes Huebschmann and the anonymous referee for many useful comments.

1 Lie Rinehart (Bi-)Algebras

Let \mathbf{A} be a unitary commutative algebra over $K = \mathbb{R}$ or \mathbb{C} . A derivation of \mathbf{A} is a K -liner map, $\delta : \mathbf{A} \rightarrow \mathbf{A}$, satisfying the Leibnitz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathbf{A}$. The \mathbf{A} -module $\text{Der}(\mathbf{A})$ of all derivations of \mathbf{A} is a K -Lie algebrac under the commutator $[\delta, \lambda] \triangleq \delta \circ \lambda - \lambda \circ \delta$. In particular, for the ring of polynomials $K[t]$, any $\delta \in \text{Der}K[t]$ has the form $\delta = f \frac{d}{dt}$, where f is uniquely determined by $f = \delta(t)$. Under this expression, we have

$$[f \frac{d}{dt}, g \frac{d}{dt}] = (fg' - f'g) \frac{d}{dt},$$

where $f' = \frac{d}{dt}f$. Thus, with the Lie bracket: $[f, g] = fg' - f'g$ for all $f, g \in K[t]$, we can identify $K[t]$ with $\text{Der}K[t]$.

Definition 1.1. A Lie Rinehart algebra is a pair (\mathbf{A}, \mathbf{E}) where \mathbf{E} is both an \mathbf{A} -module as and a K -Lie algebra such that

- 1) there is a Lie algebra morphism $\theta: \mathbf{E} \rightarrow \text{Der}(\mathbf{A})$ (called the anchor of \mathbf{E}) which is also a morphism of \mathbf{A} -modules;
- 2) $[X_1, aX_2] = a[X_1, X_2] + \theta(X_1)(a)X_2, \forall X_i \in \mathbf{E}, a \in \mathbf{A}.$

In case that \mathbf{A} is fixed we just say that \mathbf{E} is a Lie Rinehart algebra.

• Crossed Products

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a K -Lie algebra. We denote the \mathbf{A} -module $\mathbf{A} \otimes_K \mathfrak{g}$ by $\mathbf{A} \otimes \mathfrak{g}$ and write an element $a \otimes X$ as aX . An action of \mathfrak{g} on \mathbf{A} means a Lie algebra morphism $\theta: \mathfrak{g} \rightarrow \text{Der}(\mathbf{A})$. By the same symbol $\theta: \mathbf{A} \otimes \mathfrak{g} \rightarrow \text{Der}(\mathbf{A})$ to denote the \mathbf{A} -module morphism extended from this action, then we have an induced bracket defined on $\mathbf{A} \otimes \mathfrak{g}$:

$$[aX, bY] \triangleq ab[X, Y] + a(\theta(X)b)Y - b(\theta(Y)a)X, \quad \forall a, b \in \mathbf{A}, X, Y \in \mathfrak{g}$$

such that $\mathbf{A} \otimes \mathfrak{g}$ is a Lie Rinehart algebra.

Definition 1.2 ([23]). *The triple $(\mathbf{A} \otimes \mathfrak{g}, [\cdot, \cdot], \theta)$ is called a crossed product, generated by \mathfrak{g} via the action θ , which is said to be nontrivial (resp. trivial) if θ is nontrivial (resp. trivial).*

The reader is recommended to compare with the so-called “action Lie algebroids” or “transformation algebroids” [2, 19, 24] to understand the geometric background of crossed products.

To introduce a special kind of crossed products, we define firstly the notion of a derivation of an \mathbf{A} -Lie algebra.

Definition 1.3. *Let $(L, [\cdot, \cdot]_L)$ be an \mathbf{A} -Lie algebra, i.e., a Lie Rinehart algebra with a trivial anchor. A derivation of L is a pair (\mathcal{D}, δ) , where $\mathcal{D}: L \rightarrow L$ is a K -linear operator, $\delta \in \text{Der}(\mathbf{A})$ and they satisfy the conditions:*

$$\begin{aligned} \mathcal{D}[l_1, l_2]_L &= [\mathcal{D}l_1, l_2]_L + [l_1, \mathcal{D}l_2]_L, \quad \forall l_1, l_2 \in L, \\ \mathcal{D}(al) &= \delta(a)l + a\mathcal{D}l, \quad \forall a \in \mathbf{A}, l \in L. \end{aligned}$$

Proposition 1.4. *For a derivation (\mathcal{D}, δ) as above, $\mathbf{A} \oplus L$ has a natural crossed product structure given by:*

$$\begin{aligned} \theta(a, l) &= a\delta, \\ [(a_1, l_1), (a_2, l_2)] &= (a_1\delta(a_2) - a_2\delta(a_1), [l_1, l_2]_L + a_1\mathcal{D}l_2 - a_2\mathcal{D}l_1), \end{aligned}$$

for all $(a, l), (a_i, l_i) \in \mathbf{A} \oplus L$.

We will call the Lie Rinehart algebra constructed in this way an *extended crossed product* of L via a derivation (\mathcal{D}, δ) , and it will be denoted by $\mathbf{A} \ltimes_{(\mathcal{D}, \delta)} L$. It is said to be *nontrivial*, if δ is not zero.

Definition 1.5. Let $(\mathcal{D}_i, \delta_i)$ ($i = 1, 2$) be two derivations of \mathbf{A} -Lie algebras $(L_i, [\cdot, \cdot]_i)$ respectively. They are said to be equivalent, written $(\mathcal{D}_1, \delta_1; L_1) \sim (\mathcal{D}_2, \delta_2; L_2)$ if there exists an \mathbf{A} -Lie algebra isomorphism $\Phi : L_1 \rightarrow L_2$, an invertible element $a_0 \in \mathbf{A}$ and $l_0 \in L_2$ such that

$$\begin{cases} \mathcal{D}_1 &= a_0 \Phi^{-1} \mathcal{D}_2 \Phi + [\Phi^{-1}(l_0), \cdot]_1, \\ \delta_1 &= a_0 \delta_2. \end{cases}$$

It is obvious that “ \sim ” is equivalence relation and the following proposition is easy to be verified.

Proposition 1.6. Assume that \mathbf{A} has no zero-divisors. Let $(\mathcal{D}_i, \delta_i)$ ($i = 1, 2$) be respectively derivations of the \mathbf{A} -Lie algebras $(L_i, [\cdot, \cdot]_i)$ and assume that $\delta_i \neq 0$. Then $\mathbf{A} \ltimes_{(\mathcal{D}_i, \delta_i)} L_i$ ($i = 1, 2$) are isomorphic if and only if $(\mathcal{D}_1, \delta_1; L_1) \sim (\mathcal{D}_2, \delta_2; L_2)$.

• The Schouten bracket and Gerstenhaber algebras

A Gerstenhaber algebra consists of a triple $(A = \sum_{i \in \mathbb{Z}} A^i, \wedge, [\cdot, \cdot])$ such that (A, \wedge) is a graded commutative associative algebra over K and $(A = \sum_{i \in \mathbb{Z}} A^{(i)}, [\cdot, \cdot])$ is a graded Lie algebra, where $A^{(i)} = A^{i+1}$, such that $[a, \cdot]$ is a derivation with respect to \wedge of degree $(i - 1)$ for any $a \in A^i$.

It is shown in [15] that a Lie Rinehart algebra E corresponds a Schouten algebra $\wedge_{\mathbf{A}}^{\bullet} E$, which is, in fact, a Gerstenhaber algebra [16] (see also Theorem 5 in [7]). The Schouten bracket is a K -bilinear bracket $[\cdot, \cdot]: \wedge_{\mathbf{A}}^k E \times \wedge_{\mathbf{A}}^l E \rightarrow \wedge_{\mathbf{A}}^{k+l-1} E$ such that $(\wedge_{\mathbf{A}}^{\bullet} E, \wedge_{\mathbf{A}}, [\cdot, \cdot])$ forms a Gerstenhaber algebra such that:

- a. It coincides with the original Lie bracket on E .
- b. $[x, f] = \theta(x)f$, $\forall f \in \mathbf{A}$, $x \in E$.
- c. It is a derivation in the graded sense, i.e.,

$$[x, y \wedge_{\mathbf{A}} z] = [x, y] \wedge_{\mathbf{A}} z + (-1)^{(|x|-1)|y|} y \wedge_{\mathbf{A}} [x, z],$$

where $x \in \wedge_{\mathbf{A}}^{|x|} E$, $y \in \wedge_{\mathbf{A}}^{|y|} E$, $z \in \wedge_{\mathbf{A}}^{\bullet} E$.

Conversely, the axioms of a Gerstenhaber algebra $(A = \sum_{i \in \mathbb{Z}} A^i, \wedge, [\cdot, \cdot])$ naturally imply that (A^0, A^1) is a Lie Rinehart algebra, such that $\theta(x)f = [x, f]$, for each $x \in A^1$ and $f \in A^0$.

Let E be a Lie Rinehart algebra and F an \mathbf{A} -module. By saying a representation of E on F , we mean an \mathbf{A} -map: $E \times F \rightarrow F$, $x \times s \mapsto x.s$, satisfying the following axioms:

$$\begin{aligned} (fx).s &= f(x.s); \\ x.(fs) &= f(x.s) + \theta(x)(f)s; \\ x.(y.s) - y.(x.s) &= [x, y].s, \quad \forall s \in F, x, y \in E, f \in \mathbf{A}. \end{aligned}$$

An \mathbf{A} -map $\Omega : E \rightarrow F$ is called a 1-cocycle, if

$$\Omega[x, y] = x.\Omega(y) - y.\Omega(x), \quad \forall x, y \in E. \quad (1)$$

For example, let $L = \text{Ker}\theta$, which is clearly an \mathbf{A} -module, as well as an ideal. We define the *adjoint representation* of E on L : (or on $\wedge_{\mathbf{A}}^k L$ in the sense of the Schouten bracket, for some $k \geq 2$)

$$x.l \triangleq [x, l], \quad \forall x \in E, l \in L \text{ (or } \wedge_{\mathbf{A}}^k L \text{)}.$$

• Lie Rinehart bialgebras

A differential Gerstenhaber algebra is a Gerstenhaber algebra equipped with a derivative operator σ , called the differential, which is of degree 1 and square zero. It is called a strong differential Gerstenhaber algebra if σ is also a derivation of the graded Lie bracket [28]. We recall a similar concept, namely the Lie Rinehart bialgebras, first introduced by Huebschmann [13].

A graded operator (of degree 1) on $\wedge_{\mathbf{A}}^{\bullet} E$ is a K -linear operator $\sigma : \wedge_{\mathbf{A}}^k E \rightarrow \wedge_{\mathbf{A}}^{k+1} E$ satisfying

$$\sigma(x \wedge_{\mathbf{A}} y) = \sigma x \wedge_{\mathbf{A}} y + (-1)^{|x|} x \wedge_{\mathbf{A}} \sigma y, \quad \forall x \in \wedge_{\mathbf{A}}^{|x|} E, y \in \wedge_{\mathbf{A}}^{\bullet} E.$$

Let σ be a graded operator, then it induces two structures on $E_{\mathbf{A}}^* = \text{Hom}_{\mathbf{A}}(E, \mathbf{A})$, the σ -anchor θ_{σ} and σ -bracket $[\cdot, \cdot]_{\sigma}$, such that

$$\begin{aligned} \theta_{\sigma}(\xi)f &= \langle \sigma f, \xi \rangle \\ \langle [\xi, \eta]_{\sigma}, x \rangle &= -\langle \sigma x, \xi \wedge_{\mathbf{A}} \eta \rangle + \theta_{\sigma}(\xi) \langle x, \eta \rangle - \theta_{\sigma}(\eta) \langle x, \xi \rangle. \end{aligned}$$

The following proposition is a fundamental criterion.

Proposition 1.7 ([15]). *Equipped with the two structures given by the graded operator σ , $E_{\mathbf{A}}^*$ is a Lie Rinehart algebra if and only if $\sigma^2 = 0$.*

Definition 1.8. *Let $\sigma : \wedge_{\mathbf{A}}^k E \rightarrow \wedge_{\mathbf{A}}^{k+1} E$ be a graded operator of degree 1. If $\sigma^2 = 0$ and*

$$\sigma[x, y] = [\sigma x, y] + (-1)^{(|x|+1)} [x, \sigma y], \quad \forall x \in \wedge_{\mathbf{A}}^{|x|} E, y \in \wedge_{\mathbf{A}}^{\bullet} E, \quad (2)$$

(E, σ) is called a Lie Rinehart bialgebra.

Thus for a strong differential Gerstenhaber algebra $(\wedge_{\mathbf{A}}^{\bullet} E, \wedge_{\mathbf{A}}, [\cdot, \cdot])$ with differential σ , (E, σ) is naturally a Lie Rinehart bialgebra. We omit the proofs of the following three propositions since they are straightforward.

Proposition 1.9. *Let σ be a graded operator of degree 1 which is also square zero. Suppose that*

$$\sigma[x, y] = [\sigma x, y] + [x, \sigma y], \quad \forall x, y \in E. \quad (3)$$

1) If E is nondegenerate, i.e., for $x \in E$,

$$x \wedge_{\mathbf{A}} y = 0, \quad \forall y \in E \quad \text{implies} \quad x = 0,$$

then

$$\sigma[x, f] = [\sigma x, f] + [x, \sigma f], \quad \forall x \in E, \quad f \in \mathbf{A}.$$

2) If E is nondegenerate and faithful, i.e., for $a \in \mathbf{A}$,

$$ax = 0, \quad \forall x \in E \quad \text{implies} \quad a = 0,$$

then (2) holds.

3) Define a bracket

$$\{a, b\} = [\sigma a, b], \quad \forall a, b \in \mathbf{A}.$$

If E is nondegenerate, then the algebra $(\mathbf{A}, \{\cdot, \cdot\})$ is a Leibnitz algebra. Moreover, the bracket is skew-symmetric if E is faithful, then \mathbf{A} is a Poisson algebra in this case.

4) If E is nondegenerate and faithful, then $(\wedge_{\mathbf{A}}^{\bullet} E, \wedge_{\mathbf{A}}, [\cdot, \cdot])$ is a strong differential Gerstenhaber algebra with the differential σ .

We are going to discover further properties of Lie Rinehart bialgebras with some additional conditions. First we review a special kind of Lie Rinehart bialgebra which generalized the method that a Poisson tensor π on a manifold gives a π -bracket for the 1-forms. This method is also referred as the dualization of a Lie Rinehart algebra (see Kosmann and Magri's definition in [15]).

If Λ is a bivector of E , i.e., $\Lambda \in \wedge_{\mathbf{A}}^2 E$, then clearly the operator $\sigma = [\Lambda, \cdot]$ satisfies condition (2). When $\sigma^2 = 0$, or equivalently, $[[\Lambda, \Lambda], \cdot] = 0$, then (E, σ) is a Lie Rinehart bialgebra. For objects of this type, we call them *coboundary* (or exact) ones [17]. Especially, when $[\Lambda, \Lambda] = 0$, we call Λ a Poisson bivector and $(E, [\Lambda, \cdot])$ a *triangular* Lie Rinehart bialgebra. Several examples will be given later after Theorem 4.1.

Let E be a Lie Rinehart algebra and Λ a bivector. We will use the symbol Λ^{\sharp} to denote the contraction map $E_{\mathbf{A}}^* \rightarrow E$, defined by $\Lambda^{\sharp}(\phi) = \phi \lrcorner \Lambda$ (this is legal since Λ can be expressed as a finite sum $\sum a_i \wedge b_i$, and thus $\phi \lrcorner \Lambda = \phi(a_i)b_i - \phi(b_i)a_i$). The operation

$$[\phi, \psi]_{\Lambda} = d \langle \Lambda, \phi \wedge \psi \rangle + \Lambda^{\sharp}(\phi) \lrcorner d\psi - \Lambda^{\sharp}(\psi) \lrcorner d\phi,$$

for $\phi, \psi \in E_{\mathbf{A}}^*$, is called the Λ -bracket on $E_{\mathbf{A}}^*$. Equivalent expressions are given as follows.

$$\begin{aligned} & \langle [\phi, \psi]_{\Lambda}, x \rangle \\ &= \langle [x, \Lambda], \phi \wedge \psi \rangle + \theta(\Lambda^{\sharp}(\phi)) \langle \psi, x \rangle - \theta(\Lambda^{\sharp}(\psi)) \langle \phi, x \rangle, \end{aligned} \quad (4)$$

for any two $\phi, \psi \in E_{\mathbf{A}}^*$. We omit the proof of these relations. By (4), one is able to get the following proposition.

Proposition 1.10. *Let E be a Lie Rinehart algebra and Λ a bivector of E such that $[[\Lambda, \Lambda], \cdot] = 0$. Then, for the Lie Rinehart bialgebra $(E, [\Lambda, \cdot])$, the corresponding Lie Rinehart algebra E_{Λ}^* given by Proposition 1.7, has the anchor map $\theta_* = \theta \circ \Lambda^\sharp$ and bracket $[\cdot, \cdot]_* = [\cdot, \cdot]_{\Lambda}$.*

2 Classification of $K[t]$ -Crossed Products

The algebra we study in this section is always assumed to be the ring of polynomials $\mathbf{A} = K[t]$ and \mathfrak{g} is assumed to be a finite dimensional Lie algebra over K . First recall that any Lie algebra \mathfrak{g} admits a unique maximal solvable ideal of \mathfrak{g} , denoted by $\mathfrak{J}(\mathfrak{g})$, and called the **radical**, or Jacobson root. The famous Levi decomposition of a Lie algebra is expressed as $\mathfrak{g} = \mathfrak{J}(\mathfrak{g}) \rtimes \mathfrak{m}$, where \mathfrak{m} is a semisimple Lie subalgebra (known as the **Levi subalgebra** of \mathfrak{g} , which is not necessarily unique [27]).

Next we quote the following result in [4] as the first step of the classification of crossed products $K[t] \otimes \mathfrak{g}$, where all actions of an arbitrary Lie algebra \mathfrak{g} on $K[t]$ are classified into three types according to $\text{Rank}(\theta)$, i.e., the dimension of $\text{Im}(\theta)$.

Theorem 2.1. [4] *Let \mathfrak{g} be a Lie algebra. Let $\mathfrak{J}(\mathfrak{g})$ be its radical and \mathfrak{m} a Levi subalgebra. If $\theta : \mathfrak{g} \rightarrow K[t]$ is a nontrivial action, then $\text{Rank}(\theta) \leq 3$ and the action has following three possible types:*

- Type 1: $\text{Rank}(\theta) = 1$. In this case, $\theta|_{\mathfrak{m}} = 0$, and there exists a polynomial $h \in K[t]$ and a linear function $\lambda \in \mathfrak{g}^*$ (both nonzero), such that

$$\theta(X) = \lambda(X)h, \quad \forall X \in \mathfrak{g}. \quad (5)$$

- Type 2: $\text{Rank}(\theta) = 2$. In this case, $\theta|_{\mathfrak{m}} = 0$, and there exist a non-negative integer $m \neq 1$, a constant $b \in K$, and two linearly independent $\lambda, \mu \in \mathfrak{g}^*$, such that

$$\theta(X) = \lambda(X)(t+b)^m + \mu(X)(t+b), \quad \forall X \in \mathfrak{g}. \quad (6)$$

- Type 3: $\text{Rank}(\theta) = 3$. In this case, one is able to decompose $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{m}_0$, where $\mathfrak{s} \cong \mathfrak{sl}(2, K)$, \mathfrak{m}_0 is a semisimple Lie subalgebra such that $\text{Ker}(\theta) = \mathfrak{J}(\mathfrak{g}) \rtimes \mathfrak{m}_0$. Moreover, one is able to find a standard base $X_0, X_1, X_2 \in \mathfrak{s}$, such that $\theta(X_0) = 1, \theta(X_1) = t, \theta(X_2) = t^2$.

As a standard base of $\mathfrak{sl}(2, K) \subset \mathfrak{gl}(2, K)$, three vectors

$$\mathbf{E}_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{E}_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

are related by $[\mathbf{E}_1, \mathbf{E}_2] = \mathbf{E}_2, [\mathbf{E}_1, \mathbf{E}_0] = -\mathbf{E}_0, [\mathbf{E}_2, \mathbf{E}_0] = -2\mathbf{E}_1$.

Example 2.2. We define a special θ_s (called the standard action) of $\mathfrak{sl}(2, K)$ on $K[t]$:

$$\begin{cases} \theta_s(\mathbf{E}_1) &= t, \\ \theta_s(\mathbf{E}_2) &= t^2, \\ \theta_s(\mathbf{E}_0) &= 1. \end{cases}$$

It is typically an action of Type 3.

Proposition 2.3. With the same assumptions as in Theorem 2.1, then

- 1) \mathfrak{g} admits an action of Type 1 if and only if $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. In this case, (5) defines an action if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \text{Ker}\lambda$.
- 2) \mathfrak{g} admits an action of Type 2 if and only if there exist two independent vectors $x_0, y_0 \in \mathfrak{g}$ and an ideal $S \subset \mathfrak{g}$ such that
 - 2.1) $\mathfrak{g} = S \oplus \langle x_0 \rangle \oplus \langle y_0 \rangle$;
 - 2.2) $[\mathfrak{g}, \mathfrak{g}] \subset S \oplus \langle x_0 \rangle$;
 - 2.3) $[x_0, y_0] + (m-1)x_0 \in S$, for some nonnegative integer $m \neq 1$.

In this case, by setting

$$\lambda|_{S \oplus \langle y_0 \rangle} = 0, \quad \text{and} \quad \lambda(x_0) = 1,$$

$$\mu|_{S \oplus \langle x_0 \rangle} = 0, \quad \text{and} \quad \mu(y_0) = 1,$$

Equation (6) defines an action.

- 3) \mathfrak{g} admits an action of Type 3 if and only if \mathfrak{g} is not solvable and the Levi subalgebra of \mathfrak{g} admits $\mathfrak{sl}(2, K)$ as an ideal.

Proof. The statements (1) and (3) are direct consequences of Theorem 2.1. We elaborate on (2). In fact, a simple calculation shows that, the map θ defined by (6) is an action if and only if for $X, Y \in \mathfrak{g}$,

$$\begin{aligned} \mu([X, Y]) &= 0, \\ \lambda([X, Y]) &= (1-m)(\lambda(X)\mu(Y) - \lambda(Y)\mu(X)). \end{aligned}$$

In this case, $S = \text{Ker}\mu \cap \text{Ker}\lambda$ is an ideal. We select $x_0, y_0 \in \mathfrak{g} - S$, satisfying $\lambda(x_0) = \mu(y_0) = 1$, $\lambda(y_0) = \mu(x_0) = 0$. Then the three conditions 2.1) \sim 2.3) are satisfied. The converse is also obvious. \square

Recall Proposition 1.6 which gives a classification of nontrivial extended crossed products using the data (\mathcal{D}, δ) . The following theorem claims that any nontrivial crossed product $K[t] \otimes \mathfrak{g}$ can be realized as an extended one. Thus we obtain a classification of such crossed products.

Theorem 2.4. For any nontrivial action $\theta : \mathfrak{g} \rightarrow K[t]$, the corresponding crossed product $K[t] \otimes \mathfrak{g}$ is isomorphic to an extended crossed product $K[t] \ltimes_{(\mathcal{D}, \delta)} L$. The data L , \mathcal{D} and δ are respectively specified as follows:

- 1) if θ is of Type 1 defined by (5), then $L = K[t] \otimes \text{Ker}\lambda$, $\mathcal{D} = [x_0, \cdot]$. $\delta = h \frac{d}{dt}$, where $x_0 \in \mathfrak{g}$ satisfies $\lambda(x_0) = 1$;
- 2) if θ is of Type 2 defined by (6), then $L = K[t] \otimes (S \oplus \langle x_0 - (t+b)^{m-1}y_0 \rangle)$, $\mathcal{D} = [y_0, \cdot]$, $\delta = (t+b) \frac{d}{dt}$, where x_0, y_0 and S are specified by (2) of Proposition 2.3;
- 3) if θ is of Type 3, then

$$L = K[t] \otimes (\mathfrak{J}(\mathfrak{g}) \oplus \mathfrak{m}_0 \oplus \langle X_2 - tX_1, X_1 - tX_0 \rangle),$$

$\mathcal{D} = [X_0, \cdot]$ and $\delta = \frac{d}{dt}$, where X_0, X_1, X_2 is a basis of $\mathfrak{s} \cong \mathfrak{sl}(2, K)$ declared in (3) of Theorem 2.1.

Proof. 1) By (1) of Proposition 2.3, $[\mathfrak{g}, \mathfrak{g}] \subset \text{Ker}\lambda$, $\mathcal{D}(L) \subset L$, and hence we have the conclusion.

2) One only need to check that $\mathcal{D}(L) \subset L$.

3) By Theorem 2.1, we conclude that the Levi subalgebra $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{m}_0$ and θ must be trivial on $\mathfrak{J}(\mathfrak{g}) \rtimes \mathfrak{m}_0$. Moreover, $[X_1, X_2] = X_2$, $[X_1, X_0] = -X_0$, $[X_2, X_0] = -2X_1$. Thus, the crossed product $K[t] \otimes \mathfrak{g}$ is spanned (over $K[t]$) by:

$$X_0, \quad A = X_2 - tX_1, \quad B = X_1 - tX_0 \quad \text{and elements in } \mathfrak{J}(\mathfrak{g}) \rtimes \mathfrak{m}_0.$$

Clearly, $L = K[t] \otimes (\mathfrak{J}(\mathfrak{g}) \oplus \mathfrak{m}_0 \oplus \langle A, B \rangle)$ is the kernel of $\theta: K[t] \otimes \mathfrak{g} \rightarrow K[t]$. And $(\mathcal{D} = [X_0, \cdot], \frac{d}{dt})$ is a derivation of L . In this way, the Lie Rinehart algebra $K[t] \otimes \mathfrak{g} \cong K[t] \rtimes_{(\mathcal{D}, \frac{d}{dt})} L$ by identifying $X_0 = 1_{K[t]}$. \square

Consider $K(t)$, the fractional field of $K[t]$ and treat $K(t)$ as a $K[t]$ -module, as well as for $K(t) \otimes \mathfrak{g}$. We also have $\text{Der}K(t) \cong K(t)$. Note that $K(t) \otimes \mathfrak{g}$ is a $K(t)$ -crossed product in an obvious sense.

Proposition 2.5. *Let θ be a nontrivial action of \mathfrak{g} on $K[t]$. Then,*

- 1) *one can find a $K[t]$ -module map $\gamma: K[t] \rightarrow K(t) \otimes \mathfrak{g}$ such that $\theta \circ \gamma = \text{Id}_{K[t]}$;*
- 2) *the corresponding crossed product is of extended type, i.e., for L being the kernel of $\theta: K(t) \otimes \mathfrak{g} \rightarrow K(t)$, and $(\mathcal{D}, \frac{d}{dt})$ a derivation of L , one has $K(t) \otimes \mathfrak{g} \cong K(t) \rtimes_{(\mathcal{D}, \frac{d}{dt})} L$.*
- 3) *if θ is of Type 3, γ takes values in $K[t] \otimes \mathfrak{g}$.*

Proof. We directly construct γ . If θ is of Type 1, then we can find an $X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$ such that $\lambda(X) = 1$. In this case, we set $\gamma(1) = \frac{1}{h}X$. If θ is of Type 2, then one can also find an $X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$ such that $\mu(X) = 1$. In this case, we set

$$\gamma(1) = \frac{1}{\lambda(X)(t+b)^m + (t+b)}X,$$

whence the first statement. If θ is of Type 3, we set $\gamma(1) = X_0$, which was declared in (3) of Theorem 2.1. The second statement is a direct consequence of the first one. \square

We finally notice the interesting fact that by Proposition 2.3, for any nontrivial action θ of a semisimple \mathfrak{g} on $K[t]$, the effective part of this action merely comes from $\mathfrak{sl}(2, K)$ and θ must be of Type 3.

Theorem 2.6. *Let $E = K[t] \otimes \mathfrak{sl}(2, K)$ be a crossed product, with the structure coming from a nontrivial action $\theta : \mathfrak{sl}(2, K) \rightarrow K[t]$. What ever θ is chosen, all these E are isomorphic to each other.*

Proof. It is not hard to see the following fact: for any two nontrivial morphisms $\theta_1, \theta_2 : \mathfrak{sl}(2, K) \rightarrow K[t]$, there exists an automorphism Π of $\mathfrak{sl}(2, K)$ such that $\theta_1 = \theta_2 \circ \Pi$. We write E^1, E^2 to indicate the two crossed products. Now we define an isomorphism $\bar{\Pi}$ from E^1 to E^2 , which maps fX to $f\Pi(X)$ ($f \in K[t]$, $X \in \mathfrak{sl}(2, K)$). The second statement is a direct consequence of the first one. \square

In general, for a semisimple Lie algebra \mathfrak{g} which admits an action of Type 3, it has a simple ideal isomorphic to $\mathfrak{sl}(2, K)$, so that $\mathfrak{g} \cong \mathfrak{sl}(2, K) \oplus \mathfrak{m}_0$, for some semisimple ideal \mathfrak{m}_0 . Any nontrivial crossed product $K[t] \otimes \mathfrak{g}$ is isomorphic to $K[t] \otimes \mathfrak{sl}(2, K) \oplus K[t] \otimes \mathfrak{m}_0$, where $\mathfrak{sl}(2, K)$ has the standard action and \mathfrak{m}_0 has the trivial action on $K[t]$.

3 Lie Rinehart Coalgebras for Crossed Products

It seems that for a non-semisimple Lie algebra \mathfrak{g} , the structure of a Lie Rinehart bialgebra of crossed product $(K[t] \otimes \mathfrak{g}, d_*)$ is quite complicated. We shall discuss the situation that \mathfrak{g} is semisimple in the next section. However, we can still say something about the operator d_* .

In [3], we proved that for a transitive Lie algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$, the structure of any Lie bialgebroid (\mathcal{A}, d_*) can be characterized by a bisection $\Lambda \in \Gamma(\wedge^2 \mathcal{A})$ and a Lie algebroid 1-cocycle, $\Omega : \mathcal{A} \rightarrow \wedge^2 L$, with respect to the adjoint representation of \mathcal{A} on $\wedge^2 L$, where $L = \text{Ker} \rho$ is the isotropic bundle of \mathcal{A} . Moreover, such a pair is unique up to a gauge term in $\Gamma(\wedge^2 L)$ and the differential d_* is decomposed into

$$d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega.$$

We will show some similar results of Lie Rinehart bialgebras for $K[t] \otimes \mathfrak{g}$.

Definition 3.1. *Let \mathfrak{g} be a Lie algebra and let θ be an action of \mathfrak{g} on $K[t]$. For the crossed product $(K(t) \otimes \mathfrak{g}, [\cdot, \cdot], \theta)$, let L be the kernel of $\theta : K(t) \otimes \mathfrak{g} \rightarrow K(t)$ and $L^2 = L \wedge_{K(t)} L$. A K -linear map $\Omega : \mathfrak{g} \rightarrow L^2$ is called a 1-cocycle if*

$$\Omega[X, Y] = [\Omega(X), Y] + [X, \Omega(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

Such a 1-cocycle Ω can be extended as a derivation of the graded module Ω : $K(t) \otimes \wedge^k \mathfrak{g} \rightarrow K(t) \otimes \wedge^{k+1} \mathfrak{g}$, $k \geq 0$. For $k = 0$, it is zero. For $k = 1$, it is simply defined by $fX \mapsto f\Omega(X)$, $\forall f \in K(t)$, $X \in \mathfrak{g}$. For $k > 1$, it is defined by

$$\begin{aligned} & \Omega(u_1 \wedge_{K(t)} \cdots \wedge_{K(t)} u_k) \\ &= \sum_{i=1}^k (-1)^{i+1} u_1 \wedge_{K(t)} \cdots \wedge_{K(t)} \Omega(u_i) \wedge_{K(t)} \cdots \wedge_{K(t)} u_k, \\ & \quad \forall u_1, \dots, u_k \in K(t) \otimes \mathfrak{g}. \end{aligned}$$

One has the following formula:

$$\Omega[u, v] = [\Omega(u), v] + (-1)^{k+1} [u, \Omega(v)], \quad \forall u \in K(t) \otimes \wedge^k \mathfrak{g}, \quad v \in K(t) \otimes \wedge^l \mathfrak{g}.$$

Definition 3.2. *With the assumptions of Definition 3.1, given $\Lambda \in K(t) \otimes \wedge^2 \mathfrak{g}$ and a K -linear map $\Omega: \mathfrak{g} \rightarrow L^2$, the pair (Λ, Ω) is called compatible if Ω is a 1-cocycle and satisfies*

$$\left[\frac{1}{2}[\Lambda, \Lambda] + \Omega(\Lambda), \cdot\right] + \Omega^2 = 0, \quad \text{as a map } K(t) \otimes \wedge^2 \mathfrak{g} \rightarrow K(t) \otimes \wedge^3 \mathfrak{g}.$$

If (Λ, Ω) is compatible, then so is the pair $(\Lambda + \nu, \Omega - [\nu, \cdot])$, for any $\nu \in L^2$. Thus, two compatible pairs (Λ, Ω) and (Λ', Ω') are called equivalent, written $(\Lambda, \Omega) \sim (\Lambda', \Omega')$, if $\exists \nu \in L^2$, such that $\Lambda' = \Lambda + \nu$ and $\Omega' = \Omega - [\nu, \cdot]$.

Theorem 3.3. *Let \mathfrak{g} be a Lie algebra and let θ be a nontrivial action of \mathfrak{g} on $K[t]$. Then, there is a one-to-one correspondence between Lie Rinehart bialgebras $(K(t) \otimes \mathfrak{g}, d_*)$ and equivalence classes of compatible pairs (Λ, Ω) such that*

$$d_* = [\Lambda, \cdot] + \Omega. \quad (8)$$

We first prove the following lemma.

Lemma 3.4. *With the same assumptions, there exists some $\Lambda \in K(t) \otimes \wedge^2 \mathfrak{g}$ such that*

$$d_* f = [\Lambda, f], \quad \forall f \in K(t). \quad (9)$$

Proof. According to Proposition 2.5, we can find an element $\gamma(1) \in K(t) \otimes \mathfrak{g}$ such that $\theta \circ \gamma(1) = 1$. Then we define $\Lambda \in K(t) \otimes \wedge^2 \mathfrak{g}$ by setting

$$\Lambda \triangleq d_* t \wedge_{K(t)} \gamma(1).$$

By (3) of Proposition 1.9, there is an antisymmetric pairing satisfying

$$\{f, g\} = [d_* f, g] = f' g' [d_* t, t], \quad \forall f, g \in K(t).$$

Hence it must be zero, i.e., $[d_* f, g] = 0$. So we have

$$\begin{aligned} [\Lambda, f] &= -[d_* t, f] \gamma(1) + [\gamma(1), f] d_* t \\ &= f' \theta \circ \gamma(1) d_* t = f' d_* t = d_* f, \quad \forall f \in K(t), \end{aligned}$$

whence the result. \square

Proof of Theorem 3.3. Suppose that a Lie Rinehart bialgebra $(K(t) \otimes \mathfrak{g}, d_*)$ is given. Then with Λ given as in the above lemma, we define

$$\Omega = d_* - [\Lambda, \cdot].$$

Equation (9) implies that Ω satisfies

$$\Omega(fu) = f\Omega(u), \quad \forall u \in K(t) \otimes \mathfrak{g}, \quad f \in K(t),$$

and hence it is indeed a $K[t]$ -module morphism. To see that Ω takes values in L^2 , it suffices to prove that $[\Omega(u), f] = 0, \forall u \in K(t) \otimes \mathfrak{g}, \quad f \in K(t)$. In fact,

$$\begin{aligned} [\Omega(u), f] &= [d_*u - [\Lambda, u], f] \\ &= d_*[u, f] - [u, d_*f] - [[\Lambda, f], u] - [\Lambda, [u, f]] \\ &= 0. \end{aligned}$$

Moreover, since both d_* and $[\Lambda, \cdot]$ are derivations, so is Ω . In other words, it is a 1-cocycle. We claim that (Λ, Ω) is a compatible pair. In fact, we have the identity

$$d_*^2(u) = \left[\frac{1}{2}[\Lambda, \Lambda] + \Omega(\Lambda), u\right] + \Omega^2(u), \quad \forall u \in K(t) \otimes \mathfrak{g}.$$

Note that this equation already implies that

$$d_*^2(f) = \left[\frac{1}{2}[\Lambda, \Lambda] + \Omega(\Lambda), f\right], \quad \forall f \in K(t).$$

Therefore the compatibility of the pair is equivalent to $d_*^2 = 0$.

We show that if two compatible pairs (Λ, Ω) and (Λ', Ω') correspond to the same Lie Rinehart bialgebra $(K(t) \otimes \mathfrak{g}, d_*)$, then they are equivalent. In fact, from the assumption, we have

$$d_*f = [\Lambda, f] = [\Lambda', f], \quad \forall f \in K(t).$$

Hence $\Lambda' - \Lambda \in L^2$. We set $\Lambda' - \Lambda = \nu$ and it follows that $\Omega' = \Omega - [\nu, \cdot]$.

Conversely, given a compatible pair (Λ, Ω) , then $(K(t) \otimes \mathfrak{g}, d_*)$ is clearly a Lie Rinehart bialgebra, where the operator $d_* : K(t) \otimes \wedge^k \mathfrak{g} \rightarrow K(t) \otimes \wedge^{k+1} \mathfrak{g}$ is defined by formula (8). \square

By (3) of Proposition 2.5 and the above proof, we have the following conclusion.

Theorem 3.5. *Let θ be an action of \mathfrak{g} on $K[t]$ of Type 3. Then there is a one-to-one correspondence between Lie Rinehart bialgebras $(K[t] \otimes \mathfrak{g}, d_*)$ and equivalence classes of compatible pairs (Λ, Ω) such that $d_* = [\Lambda, \cdot] + \Omega$. Here $\Lambda \in K[t] \otimes \wedge^2 \mathfrak{g}$ and $\Omega : \mathfrak{g} \rightarrow L \wedge_{K[t]} L$, L being the kernel of $\theta : K[t] \otimes \mathfrak{g} \rightarrow K[t]$.*

Most of our paper concentrate on non-trivial actions. We now examine the case where the action is zero. Let \mathbf{A} be the ring $K[t]$ or $K(t)$. For $Y \in \mathbf{A} \otimes \mathfrak{g}$ we have an induced derivation $d_Y: \mathbf{A} \otimes \wedge^k \mathfrak{g} \rightarrow \mathbf{A} \otimes \wedge^{k+1} \mathfrak{g}$,

$$d_Y(fW) \triangleq f'Y \wedge W, \quad \forall f \in \mathbf{A}, W \in \wedge^k \mathfrak{g}.$$

Note that d_Y is not able to be written as $[\Lambda, \cdot]$, for some $\Lambda \in \mathbf{A} \otimes \wedge^2 \mathfrak{g}$.

We omit the proof of the following proposition.

Proposition 3.6. *For the trivial crossed product $\mathbf{A} \otimes \mathfrak{g}$, any Lie Rinehart bialgebra $(\mathbf{A} \otimes \mathfrak{g}, d_*)$ is uniquely determined by a 1-cocycle Ω and an element $Y \in \mathbf{A} \otimes Z(\mathfrak{g})$ such that*

$$d_* = \Omega + d_Y.$$

Moreover, Ω and Y are subject to the following conditions:

$$\Omega(Y) = 0, \quad \Omega^2 + d_Y \circ \Omega = 0, \quad \text{as a map } \mathfrak{g} \rightarrow \mathbf{A} \otimes \wedge^2 \mathfrak{g}.$$

Especially when \mathfrak{g} is semisimple, we will show that d_* is always a coboundary in the next section (Theorem 4.1).

4 Lie Rinehart Bialgebras for Semisimple $K[t]$ -Crossed Products

In this section, we study the special properties of a crossed product $K[t] \otimes \mathfrak{g}$ and a Lie Rinehart bialgebra $(K[t] \otimes \mathfrak{g}, d_*)$, where \mathfrak{g} is semisimple.

Theorem 4.1. *Let \mathfrak{g} be a semisimple Lie algebra and \mathbf{A} be an arbitrary algebra. For the trivial crossed product $\mathbf{A} \otimes \mathfrak{g}$ (i.e., $\theta = 0$), the Lie Rinehart bialgebras $(\mathbf{A} \otimes \mathfrak{g}, d_*)$ are one-to-one in correspondence with $\Lambda \in \mathbf{A} \otimes \wedge^2 \mathfrak{g}$ satisfying $[X, [\Lambda, \Lambda]] = 0, \forall X \in \mathfrak{g}$, such that $d_* = [\Lambda, \cdot]$.*

We will need the famous Whitehead's lemma which claims that for any non-trivial, finite dimensional \mathfrak{g} -module V , the cohomology groups $\mathcal{H}^1(\mathfrak{g}, V)$ and $\mathcal{H}^0(\mathfrak{g}, V)$ are both zero [14].

Proof of Theorem 4.1. Since $\mathbf{A} \otimes \mathfrak{g}$ is a freely generated \mathbf{A} -module, one is easy to see that $(\mathbf{A} \otimes \mathfrak{g}, d_*)$ becomes a Lie Rinehart bialgebra if and only if all the following three conditions hold.

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad (10)$$

$$d_*[a, Y] = [a, d_*Y] + [d_*a, Y], \quad (11)$$

$$[d_*a, b] = -[d_*b, a], \quad (12)$$

$\forall X, Y \in \mathfrak{g}, a, b \in \mathbf{A}$.

It is quite evident that $d_*(\mathfrak{g})$ is contained in a subspace $\mathfrak{D} = \sum_{i=1}^m a^i(\wedge^2 \mathfrak{g})$, $a^i \in \mathbf{A}, m \in \mathbb{N}$, which is clearly a \mathfrak{g} -module. It follows from relation (10) that

$d_*|_{\mathfrak{g}}$ is a 1-cocycle, and by the Whitehead's lemma, there exists $\Lambda \in \mathfrak{D}$ such that $d_*|_{\mathfrak{g}} = [\Lambda, \cdot]$.

Notice that for the trivial crossed product $\mathbf{A} \otimes \mathfrak{g}$, condition (11) becomes $[d_*a, Y] = 0, \forall Y \in \mathfrak{g}$. Then again by the Whitehead's lemma, we know that $d_*a = 0, \forall a \in \mathbf{A}$. Thus, $d_*(x) = [\Lambda, x]$ holds for all $x \in \mathbf{A} \otimes \mathfrak{g}$. Clearly, the condition $[X, [\Lambda, \Lambda]] = 0$ is equivalent to $d_*^2(X) = 0$.

The uniqueness of Λ is guarantied by the Whitehead's lemma. \square

In what follows, we will study a nontrivial crossed product $K[t] \otimes \mathfrak{g}$, where \mathfrak{g} is a semisimple Lie algebra possessing a nontrivial action θ on $K[t]$ of Type 3. We will classify all Lie Rinehart bialgebras $(K[t] \otimes \mathfrak{g}, d_*)$.

By Theorem 2.6, we know that \mathfrak{g} must be of the form: $\mathfrak{g} = \mathfrak{sl}(2, K) \oplus \mathfrak{l}$ where $\mathfrak{l} = \text{Ker}\theta$ is an arbitrary semisimple Lie algebra. By means of the Killing form (\cdot, \cdot) of \mathfrak{g} , one can identify \mathfrak{g}^* with \mathfrak{g} and define the Cartan 3-form Ω by

$$\Omega(X, Y, Z) = ([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g},$$

which is a Casimir element $\Omega \in \wedge^3 \mathfrak{g}$ (i.e., $[\Omega, X] = 0, \forall X \in \mathfrak{g}$). In particular, we denote the Cartan 3-form of $\mathfrak{sl}(2, K)$ by $\Omega_{sl(2)}$. Under the base $\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2$ of $\mathfrak{sl}(2, K)$ given in (7), the values of the Killing forms are determined by

$$(\mathbf{E}_1, \mathbf{E}_1) = 2, \quad (\mathbf{E}_2, \mathbf{E}_0) = (\mathbf{E}_0, \mathbf{E}_2) = -4. \quad (13)$$

Therefore, we have $\Omega_{sl(2)} = 4\mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_0$.

The Killing form is naturally extended to be a product of $K[t] \otimes \mathfrak{g}$, taking values in $K[t]$. For each $f \in K[t]$, we denote $(d_\theta f)^\# \in K[t] \otimes \mathfrak{g}$ the corresponding element for $d_\theta f \in K[t] \otimes \mathfrak{g}^*$, i.e.,

$$((d_\theta f)^\#, X) = \theta(X)f, \quad \forall X \in \mathfrak{g}.$$

We introduce a differential operator from $K[t] \otimes \wedge^k \mathfrak{g}$ to $K[t] \otimes \wedge^{k+1} \mathfrak{g}$ as follows,

$$\mathcal{D}(fX_1 \wedge \cdots \wedge X_k) = (d_\theta f)^\# \wedge X_1 \wedge \cdots \wedge X_k, \quad \forall f \in K[t], \quad \forall X_1, \dots, X_k \in \mathfrak{g}. \quad (14)$$

The operator \mathcal{D} is totally determined by $\mathcal{D}t$ since $\mathcal{D}f = f'\mathcal{D}t, \forall f \in K[t]$ and $\mathcal{D}X = 0, \forall X \in \mathfrak{g}$.

Lemma 4.2.

$$\mathcal{D}^2 t = \frac{1}{32} [\Omega_{sl(2)}, t]. \quad (15)$$

Proof. For the standard θ given in Example 2.2, we have

$$(\mathcal{D}t, \mathbf{E}_i) = \theta(\mathbf{E}_i) = t^i, \quad i = 0, 1, 2.$$

Thus the relations in (13) implies

$$\mathcal{D}t = (d_\theta t)^\# = \frac{1}{4}(2t\mathbf{E}_1 - t^2\mathbf{E}_0 - \mathbf{E}_2), \quad (16)$$

and

$$\begin{aligned}
\mathcal{D}^2 t &= \frac{1}{4}(2\mathcal{D}t \wedge \mathbf{E}_1 - 2t\mathcal{D}t \wedge \mathbf{E}_0), \\
&= \frac{1}{8}(t\mathbf{E}_2 \wedge \mathbf{E}_0 + E_1 \wedge \mathbf{E}_2 - t^2 E_1 \wedge \mathbf{E}_0) \\
&= \frac{1}{8}[\mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_0, t].
\end{aligned}$$

The latter one is exactly $\frac{1}{32}[\Omega_{sl(2)}, t]$. By Theorem 2.6, this relation must hold for any nontrivial θ . \square

Definition 4.3. *With notations above, for a constant ε and an element $\Lambda \in K[t] \otimes \wedge^2 \mathfrak{g}$, the following equation is called the ε -dynamical Yang-Baxter equation (ε -DYBE):*

$$\frac{1}{2}[\Lambda, \Lambda] + \varepsilon \mathcal{D}\Lambda + \frac{\varepsilon^2}{32}\Omega_{sl(2)} = \omega \in (\wedge^3 \mathfrak{l})^l, \quad (17)$$

where ω is an arbitrary Casimir element in $\wedge^3 \mathfrak{l}$. A solution to this equation is called an ε -dynamical r -matrix.

We remark that this notion is a special one of the notion of dynamical r -matrices coupled with Poisson manifolds introduced in [18], which is a natural generalization of the classical dynamical r -matrices of Felder [6].

The main theorem in this section is as follows:

Theorem 4.4. *For any Lie Rinehart algebra $K[t] \otimes \mathfrak{g}$, where \mathfrak{g} is a semisimple Lie algebra possessing a nontrivial action on $K[t]$, there is a one-to-one correspondence between Lie Rinehart bialgebras $(K[t] \otimes \mathfrak{g}, d_*)$ and ε -dynamical r -matrices Λ such that*

$$d_* = [\Lambda, \cdot] + \varepsilon \mathcal{D}.$$

We split the proof into several lemmas.

Lemma 4.5. *For any K -linear operator $D : \mathfrak{g} \rightarrow K[t] \otimes \wedge^2 \mathfrak{g}$ satisfying*

$$D[X, Y] = [DX, Y] + [X, DY], \quad \forall X, Y \in \mathfrak{g}, \quad (18)$$

there exists a unique $\Lambda \in K[t] \otimes \wedge^2 \mathfrak{g}$ such that $D = [\Lambda, \cdot]$.

Proof. Suppose that $D(X) = \sum_{i=0}^m t^i D_i(X)$, for each $X \in \mathfrak{g}$, where the operators $D_i : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ are all K -linear and $m \in \mathbb{N}$ is the highest degree appeared in the image of D .

Claim 1. $D_m(\mathbf{E}_1) = 0$. This is seen by comparing the highest term on both sides of the relation

$$\begin{aligned}
D(\mathbf{E}_2) &= D([\mathbf{E}_1, \mathbf{E}_2]) = [D(\mathbf{E}_1), \mathbf{E}_2] + [\mathbf{E}_1, D(\mathbf{E}_2)] \\
&= \sum_{i=0}^m t^i ([D_i(\mathbf{E}_1), \mathbf{E}_2] + [\mathbf{E}_1, D_i(\mathbf{E}_2)] + i D_i(\mathbf{E}_2)) - \sum_{i=1}^m i t^{i+1} D_i(\mathbf{E}_1).
\end{aligned}$$

Claim 2. $D_m(\mathbf{E}_0) = 0$. This comes from the relation

$$-2D(\mathbf{E}_1) = D([\mathbf{E}_2, \mathbf{E}_0]) = [D(\mathbf{E}_2), \mathbf{E}_0] + [\mathbf{E}_2, D(\mathbf{E}_0)].$$

Claim 3. $D_m(X) = 0, \forall X \in \mathfrak{l}$. This is by $[X, \mathbf{E}_2] = 0$.

Claim 4. $m \neq 1$. In fact, if $m = 1$, we suppose that $D_1(\mathbf{E}_2) = a\mathbf{E}_1 \wedge \mathbf{E}_2 + b\mathbf{E}_1 \wedge \mathbf{E}_0 + c\mathbf{E}_2 \wedge \mathbf{E}_0$, for some $a, b, c \in K$. Then comparing the two sides of the relation below *Claim 1*, one is able to get $[\mathbf{E}_1, D_1(\mathbf{E}_2)] = 0$, which implies $a = b = 0$. By comparing the relation below *Claim 2.*, one gets $[\mathbf{E}_0, D_1(\mathbf{E}_2)] = 0$, which implies $c = 0$. Thus $D_1(\mathbf{E}_2) = 0$, contradicts with our assumptions that $m = 1$ is the highest degree appeared in the image of D .

Now, we know that $D_m(\mathbf{E}_2) \neq 0$. If $m \geq 2$, we define a new operator

$$D^{(1)} \triangleq D - \frac{1}{m-1} [t^{m-1} D_m(\mathbf{E}_2), \cdot].$$

It obviously satisfies a 1-cocycle condition similar to (18). Assume that $D^{(1)} = \sum_{i=1}^n D_i^{(1)} t^i(\cdot)$, where $D_i^{(1)} : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ are all K -linear and n is the highest degree appeared in $\text{Im}(D^{(1)})$, then clearly $n \leq m$. But it is easily seen that

$$D_m^{(1)}(\mathbf{E}_1) = D_m^{(1)}(\mathbf{E}_0) = D_m^{(1)}(\mathbf{E}_2) = D_m^{(1)}(\mathfrak{l}) = 0,$$

and hence $n < m$.

In this way, the induction goes forward and it amounts to prove that $D^{(l)}$ is a coboundary, for sufficiently large $l \in \mathbb{N}$. It suffices to assume that $\text{Im}(D^{(l)}) \in \wedge^2 \mathfrak{g}$, in which case the Whitehead's Lemma is valid and this proves that D is a coboundary.

Next we show that Λ is unique, i.e., If any $\tau \in K[t] \otimes \wedge^2 \mathfrak{g}$ satisfies $[X, \tau] = 0, \forall X \in \mathfrak{g}$, then it must be zero. Write $\tau = \sum_{i=0}^m t^i \tau_i$, for some $\tau_i \in \wedge^2 \mathfrak{g}$ ($\tau_m \neq 0$), then $[\tau, \mathbf{E}_2] = 0$ becomes

$$[\tau_0, \mathbf{E}_2] + \sum_{i=1}^m t^i ([\tau_i, \mathbf{E}_2] - (i-1)\tau_{i-1}) - mt^{m+1}\tau_m = 0.$$

Thus, m must be zero, $\tau \in \wedge^2 \mathfrak{g}$. The conclusion $\tau = 0$ comes from the fact that $\mathcal{H}^0(\mathfrak{g}, \wedge^2 \mathfrak{g}) = 0$, since \mathfrak{g} is semisimple. \square

Remark 4.6. This lemma suggests that $\mathcal{H}^i(\mathfrak{g}, K[t] \otimes \wedge^2 \mathfrak{g}) = 0$ ($i = 1, 2$) is also true.

By Lemma 4.5, we know that for any 1-degree derivation d_* for the Gerstenhaber algebra $K[t] \otimes \wedge^\bullet \mathfrak{g}$, there exists a unique $\Lambda \in K[t] \otimes \wedge^2 \mathfrak{g}$ such that $d_*|_{\mathfrak{g}} = [\Lambda, \cdot]$. The next lemma gives some further information on d_* as follows.

Lemma 4.7. With notations above, then, for the following operator:

$$\mathfrak{d} \triangleq d_* - [\Lambda, \cdot] : K[t] \otimes \wedge^k \mathfrak{g} \rightarrow K[t] \otimes \wedge^{k+1} \mathfrak{g},$$

there exists a constant number ε such that $\mathfrak{d} = \varepsilon \mathcal{D}$.

Proof. Recall the three conditions listed in the proof of Theorem 4.1. In particular, $d_* = [\Lambda, \cdot] + \mathfrak{d}$, which naturally subjects to (10), is a derivation for Lie brackets if and only if d_* satisfies the other two conditions (11) and (12), i.e., $[\mathfrak{d}t, t] = \theta(\mathfrak{d}t) = 0$, and $\mathfrak{d}[X, t] = [X, \mathfrak{d}t]$, $\forall X \in \mathfrak{g}$. Thus $[X, \mathfrak{d}t] = 0$, $\forall X \in \mathfrak{l}$ and we know that $\mathfrak{d}t \in K[t] \otimes \wedge^2 \mathfrak{sl}(2, K)$. Suppose that

$$\mathfrak{d}t = \alpha \mathbf{E}_1 + \beta \mathbf{E}_2 + \gamma \mathbf{E}_0,$$

for some $\alpha, \beta, \gamma \in K[t]$. Then one obtains

$$\begin{aligned} \mathfrak{d}t &= \mathfrak{d}[\mathbf{E}_1, t] = [\mathbf{E}_1, \mathfrak{d}t] \\ &= t\alpha' \mathbf{E}_1 + (t\beta' + \beta) \mathbf{E}_2 + (t\gamma' - \gamma) \mathbf{E}_0. \end{aligned}$$

Hence $t\alpha' = \alpha$, $\beta' = 0$ and $\gamma' = 2\gamma$. So we get $\alpha = at$, $\beta = b$, $\gamma = ct^2$, where a, b, c are some constants. On the other hand, we have

$$\begin{aligned} 2t\mathfrak{d}t &= \mathfrak{d}t^2 \\ &= \mathfrak{d}[\mathbf{E}_2, t] = [\mathbf{E}_2, \mathfrak{d}t] \\ &= (t^2\alpha' - 2\gamma) \mathbf{E}_1 + (\beta' - \alpha) \mathbf{E}_2 + \gamma' \mathbf{E}_0. \end{aligned}$$

Hence we get

$$2t\alpha = t^2\alpha' - 2\gamma, \quad 2t\beta = \beta' - \alpha.$$

These two relations restrain that $a : b : c = -2 : 1 : 1$. This proves that there exists $\varepsilon \in K$ such that

$$\alpha = \frac{1}{2}\varepsilon t; \quad \beta = -\frac{1}{4}\varepsilon; \quad \gamma = -\frac{1}{4}\varepsilon t^2.$$

Then by formula (16), $\mathfrak{d}t = \varepsilon \mathcal{D}t$. □

Lemma 4.8. *For any $\Gamma \in K[t] \otimes \wedge^3 \mathfrak{g}$ satisfying $[\Gamma, X] = 0$, $\forall X \in \mathfrak{g}$, Γ must be of the form $\Gamma = k\Omega_{\mathfrak{sl}(2)} + \omega$, where k is a constant and ω is a Casimir element in $\wedge^3 \mathfrak{l}$.*

Proof. Using the same method as in the proof of Theorem 4.5, one easily gets $\Gamma \in \wedge^3 \mathfrak{g}$. So we write

$$\Gamma = \Gamma^{3,0} + \Gamma^{2,1} + \Gamma^{1,2} + \Gamma^{0,3},$$

where $\Gamma^{ij} \in \wedge^i \mathfrak{sl}(2, K) \wedge (\wedge^j \mathfrak{l})$. Clearly, $\Gamma^{0,3}$ is a Casimir element in $\wedge^3 \mathfrak{l}$, and so is $\Gamma^{3,0}$. If we write $\Gamma^{1,2} = \mathbf{E}_1 \wedge A + \mathbf{E}_2 \wedge B + \mathbf{E}_0 \wedge C$, where $A, B, C \in \wedge^2 \mathfrak{l}$, then $[\Gamma, \mathfrak{l}] = 0$ implies $[A, \mathfrak{l}] = 0$. Since \mathfrak{l} is semisimple, A must be zero. Similarly, $B = C = 0$, and $\Gamma^{1,2} = 0$. For the same reasons, $\Gamma^{2,1} = 0$. □

Proof of Theorem 4.4. By Lemma 4.7, any 1-degree derivation d_* for the Gerstenhaber algebra $K[t] \otimes \wedge^\bullet \mathfrak{g}$ has a unique decomposition

$$d_* = [\Lambda, \cdot] + \varepsilon \mathcal{D},$$

where one does not need any compatible conditions between Λ and \mathcal{D} . It is easy to check that $d_*^2 = 0$ if and only if

$$\left[\frac{1}{2}[\Lambda, \Lambda] + \varepsilon \mathcal{D}\Lambda, t\right] + \varepsilon^2 \mathcal{D}^2 t = 0 \quad (19)$$

and

$$\left[\frac{1}{2}[\Lambda, \Lambda] + \varepsilon \mathcal{D}\Lambda, X\right] = 0, \quad \forall X \in \mathfrak{g}. \quad (20)$$

Now, due to (20) and Lemma 4.8, we have

$$\frac{1}{2}[\Lambda, \Lambda] + \varepsilon \mathcal{D}\Lambda = k\Omega_{sl(2)} + \omega.$$

Moreover, by (19), (15), we obtain $k = -\frac{\varepsilon^2}{32}$. That is exactly (17). \square

Corollary 4.9. *Identifying $K[t] \otimes \mathfrak{g}^*$ with $K[t] \otimes \mathfrak{g}$ via the Killing form, for the second (dual) Lie Rinehart algebra structure on $K[t] \otimes \mathfrak{g}$, the Lie bracket and the anchor, are given by the following formulas,*

$$[x, y]_* = [x, y]_\Lambda + \varepsilon(\theta(x).y - \theta(y).x), \quad \forall x, y \in K[t] \otimes \mathfrak{g}. \quad (21)$$

and

$$\theta_* = \theta \circ (\Lambda^\sharp + \varepsilon I). \quad (22)$$

Moreover, under these two structures, $K[t] \otimes \mathfrak{g}$ is a crossed product if and only if $\Lambda \in \wedge^2 \mathfrak{g}$.

Proof. It is some straightforward calculations to verify formulas (21) and (22). In particular, for $X, Y \in \mathfrak{g}$, by relation (4), we have

$$([X, Y]_*, Z) = ([X, Y]_\Lambda, Z) = ([Z, \Lambda], X \wedge Y), \quad \forall Z \in \mathfrak{g}.$$

Thus, $[X, Y]_* \in \mathfrak{g}$, holds for all X, Y if and only if $[\mathfrak{g}, \Lambda] \in \wedge^2 \mathfrak{g}$, which simply suggests $\Lambda \in \wedge^2 \mathfrak{g}$. Only when this happens, $K[t] \otimes \mathfrak{g}$ endowed with the dual bracket and anchors, becomes a crossed product. \square

Proposition 4.10. *There exists some $\tau \in K[t] \otimes \wedge^2 \mathfrak{g}$ such that*

- 1) $\mathcal{D}t = [\tau, t]$ and τ is unique up to an element of $\wedge_{K[t]}^2 L$, where L is the kernel of $\theta : K[t] \otimes \mathfrak{g} \rightarrow K[t]$.
- 2) The operator defined by $\Omega \triangleq \mathcal{D} - [\tau, \cdot], E \rightarrow \wedge_{K[t]}^2 L$ is a 1-cocycle with respect to the adjoint representation.
- 3) One can take such $\tau \in K[t] \otimes \wedge^2 \mathfrak{sl}(2, K)$ which is also an ε -dynamical r -matrix for $\varepsilon = -1$.

Proof. We first prove (3). Let θ be the standard action. We can check that

$$\tau = -\frac{1}{4}\mathbf{E}_2 \wedge \mathbf{E}_0 + \frac{t}{2}\mathbf{E}_1 \wedge \mathbf{E}_0,$$

satisfies $\mathcal{D}t = [\tau, t]$ (c.f. Equation (16)), and it is a (-1) -dynamical r -matrix. This shows the existence of τ in (1). If $\tilde{\tau}$ is another one, then $[\tau - \tilde{\tau}, f] = 0$, $\forall f \in K[t]$ implies that $\tau - \tilde{\tau} \in \wedge_{K[t]}^2 L$. For the operator Ω defined in (2), it already satisfies condition (1). Then from $\mathcal{D}f = f'\mathcal{D}t = f'[\tau, t] = [\tau, f]$, $\forall f \in K[t]$, we get

$$\Omega(fx) = \mathcal{D}f \wedge_{K[t]} x + f\mathcal{D}(x) - [\tau, f] \wedge_{K[t]} x - f[\tau, x] = f\Omega(x), \quad \forall x \in K[t] \otimes \mathfrak{g}.$$

This shows that Ω is a $K[t]$ -linear map. \square

Now, we can determine the compatible pair declared by Theorem 3.5. In fact, the above proposition claims that for a Lie Rinehart bialgebra $(K[t] \otimes \mathfrak{sl}(2, K), d_*)$, $d_* = [\Lambda, \cdot] + \varepsilon\mathcal{D}$ can be written into the form

$$d_* = [\Lambda + \varepsilon\tau, \cdot] + \varepsilon(\mathcal{D} - [\tau, \cdot]) = [\Lambda + \varepsilon\tau, \cdot] + \varepsilon\Omega.$$

So $(\Lambda + \varepsilon\tau, \varepsilon\Omega)$ is a compatible pair.

It is seen that the case that $\mathfrak{g} = \mathfrak{sl}(2, K)$ is the most important case, which we shall now examine. Let $E = K[t] \otimes \mathfrak{sl}(2, K)$ be the Lie Rinehart algebra coming from the standard action $\theta : (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_0) \mapsto (t, t^2, 1)$. Set

$$\Lambda = u\mathbf{E}_1 \wedge \mathbf{E}_2 + v\mathbf{E}_2 \wedge \mathbf{E}_0 + w\mathbf{E}_1 \wedge \mathbf{E}_0, \quad u, v, w \in K[t].$$

By some straightforward calculations, one gets

$$[\Lambda, \Lambda] = (-v^2 - uw + \frac{1}{2}t^2[u, v] + \frac{1}{2}[v, w] + \frac{1}{2}t[u, w])\Omega_{\mathfrak{sl}(2)},$$

and

$$\mathcal{D}\Lambda = \frac{1}{16}(2tv' + w' - t^2u')\Omega_{\mathfrak{sl}(2)}.$$

Thus, we see that

$$\frac{1}{2}[\Lambda, \Lambda] + \varepsilon\mathcal{D}\Lambda = f_\varepsilon\Omega_{\mathfrak{sl}(2)},$$

where function f_ε is defined by

$$f_\varepsilon(u, v, w) \triangleq -\frac{1}{2}(v^2 + uw) + \frac{1}{4}(t^2[u, v] + [w, v] + t[u, w]) + \frac{\varepsilon}{16}(2tv' + w' - t^2u').$$

Consequently, we have

Corollary 4.11. *Let $K[t] \otimes \mathfrak{sl}(2, K)$ be the Lie Rinehart algebra with the standard action. Then*

$$\Lambda = u\mathbf{E}_1 \wedge \mathbf{E}_2 + v\mathbf{E}_2 \wedge \mathbf{E}_0 + w\mathbf{E}_1 \wedge \mathbf{E}_0$$

is an ε dynamical r -matrix if and only if $f_\varepsilon(u, v, w) = -\frac{1}{32}\varepsilon^2$, i.e.,

$$-16(v^2 + uw) + 8(t^2[u, v] + [w, v] + t[u, w]) + 2\varepsilon(2tv' + w' - t^2u') + \varepsilon^2 = 0. \quad (23)$$

Example 4.12. Assume that $u = 0$, $v = v_0$, $w(t) = w_0 t$ where v_0, w_0 are all constants, then (23) becomes $\varepsilon^2 + 2w_0\varepsilon - 8v_0(w_0 + 2v_0) = 0$. The two solutions are $\varepsilon = 4v_0$ and $\varepsilon = -2w_0 - 4v_0$.

Example 4.13. Check that $u = a_0$, $v = a_0 t + \frac{\varepsilon}{4}$, $w = -\frac{\varepsilon}{2}t - a_0 t^2$ is also a solution to (23), where a_0 is a constant.

When Λ belongs to $\wedge^2 \mathfrak{sl}(2, K)$, or u, v, w are all constants, Equation (23) becomes $v^2 + uw = \varepsilon^2/16$. So we conclude from Corollary 4.9 that

Corollary 4.14. For the Lie Rinehart bialgebra $(E = K[t] \otimes \mathfrak{sl}(2, K), d_*)$, if the induced Lie Rinehart algebra $E_{\mathbf{A}}^*$ is also a crossed product, then there exists a unique quadruple $(u, v, w, \varepsilon) \in K^4$ satisfying

$$v^2 + uw = \varepsilon^2/16, \quad (24)$$

and

$$d_* = [u\mathbf{E}_1 \wedge \mathbf{E}_2 + v\mathbf{E}_2 \wedge \mathbf{E}_0 + w\mathbf{E}_1 \wedge \mathbf{E}_0, \cdot] + \varepsilon \mathcal{D}. \quad (25)$$

Conversely, any quadruple $(u, v, w, \varepsilon) \in K^4$ satisfying (24) corresponds to a Lie Rinehart bialgebra (E, d_*) by relation (25) and $E_{\mathbf{A}}^*$ is also a crossed product.

We then consider $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1 \cong \mathfrak{g}_2 \cong \mathfrak{sl}(2, K)$. Suppose that \mathfrak{g}_1 acts nontrivially on $K[t]$ and $\text{Ker}(\theta) = \mathfrak{g}_2$. Let (7) be the standard base of \mathfrak{g}_1 , and $(\bar{\mathbf{E}}_1, \bar{\mathbf{E}}_2, \bar{\mathbf{E}}_0)$ be the standard base of \mathfrak{g}_2 . Again we assume that $\theta : (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_0) \mapsto (t, t^2, 1)$.

Example 4.15. Let $\Lambda = (t+1)\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_2 + t^2\bar{\mathbf{E}}_2 \wedge \bar{\mathbf{E}}_0 + (1-t)\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_0$ be an element in $K[t] \otimes \wedge^2 \mathfrak{g}_2$. Then, Λ is a 0-dynamical r -matrix.

Suppose that a bisection of $K[t] \otimes \wedge^2 \mathfrak{g}$ given by

$$\Lambda = a\mathbf{E}_1 \wedge \mathbf{E}_2 + b\mathbf{E}_2 \wedge \mathbf{E}_0 + c\mathbf{E}_1 \wedge \mathbf{E}_0 + u\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_2 + v\bar{\mathbf{E}}_2 \wedge \bar{\mathbf{E}}_0 + w\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_0,$$

where $a, b, c, u, v, w \in K[t]$. Then

$$\begin{aligned} & \frac{1}{2}[\Lambda, \Lambda] + \varepsilon \mathcal{D}\Lambda \\ &= f_\varepsilon(a, b, c)\Omega_{\mathfrak{sl}(2)} - \frac{1}{2}(v^2 + uw)\Omega_{\mathfrak{sl}(2)}^- \\ & \quad + ((at^2 + c + \frac{1}{2}\varepsilon t)\mathbf{E}_1 + ((b - \frac{1}{4}\varepsilon) - at)\mathbf{E}_2 - ((b + \frac{1}{4}\varepsilon)t^2 + ct)\mathbf{E}_0) \\ & \quad \wedge (u'\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_2 + v'\bar{\mathbf{E}}_2 \wedge \bar{\mathbf{E}}_0 + w'\bar{\mathbf{E}}_1 \wedge \bar{\mathbf{E}}_0). \end{aligned}$$

Hence, Λ is a solution to the ε -DYBE if and only if $f_\varepsilon(a, b, c) = -\frac{1}{32}\varepsilon^2$, $v^2 + uw$ is a constant and

$$\begin{cases} at^2 + c + \frac{1}{2}\varepsilon t &= 0, \\ (b - \frac{1}{4}\varepsilon) - at &= 0, \\ (b + \frac{1}{4}\varepsilon)t^2 + ct &= 0. \end{cases}$$

There are many solutions to these conditions. For an example, $a = a_0$, $b = a_0 t + \frac{\varepsilon}{4}$, $c = -\frac{\varepsilon}{2}t - a_0 t^2$ ($a_0 \in K$), $u = t + 1$, $v = t^2$, $w = t - 1$.

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